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Effect of Multipath on the Height-Finding Capability of Fixed-Reflector Radar Systems

Part 4: Effect of System Noise

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We have studied the effect of noise and clutter on the elevation accuracy of the reflector radar system described in RADC/TR-76-215. ~~We have~~ found that the elevation error can be separated into two contributions: one due to multipath alone and the other, which is a combination of the effects of noise, multipath, and ground clutter.

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Contents

Illustrations

Table

Effect of Multipath on the Height-Finding Capability of Fixed-Reflector Radar Systems
Part 4: Effect of System Noise

1. INTRODUCTION

In previous reports[1-3] we have studied the effects of multipath and the CW-396A radome on the height-finding capabilities of an air search radar. In this portion we will consider the effects of system noise on its operation. A summary of our results on noise effects is contained in Table 1.

(Received for publication 26 January 1977)

1. Fante, R.L., Franchi, P.R., and Taylor, R.L. (1976) Effect of Multipath on the Height-Finding Capabilities of a Fixed-Reflector Radar System – Part 1: Analysis, RADC-TR-76-160.
2. Fante, R.L., Franchi, P.R., and Taylor, R.L. (1976) Effect of Multipath on the Height-Finding Capability of Fixed-Reflector Radar Systems – Part 2: Application to an Air Search Radar System, RADC-TR-76-215.
3. Fante, R.L., Franchi, P.R., and Taylor, R.L. (1976) Effect of Multipath on the Height-Finding Capabilities of a Fixed-Reflector Radar System – Part 3: Effect of Radome, RADC-TR-76-370.

Table 1. Summary of Important Results for Target at Elevation

(a) Multipath Absent (No Clutter)

$$\text{angular error} = \sigma_\theta = \frac{8.7\left\{1+\left[\frac{f_1(\theta)}{f_o(\theta)}\right]^2\right\}^{1/2}}{\left[2\left(\frac{S}{N}\right)_1\right]^{1/2}\frac{d}{d\theta}\left\{10\log_{10}\left[\frac{f_1(\theta)}{f_o(\theta)}\right]^2\right\}}.$$

(b) Multipath Present (No Clutter)

$$\text{angular error} = \sigma_\theta + \left|\langle\Delta\theta\rangle\right|$$

$$= \frac{\left|\log_{10}\left(\frac{M_1}{M_o}\right)\right| + 8.7\left[2\left(\frac{S}{N}\right)_{M1}\right]^{-1/2}\left[1+\frac{M_1 f_1^2(\theta)}{M_o f_o^2(\theta)}\right]^{1/2}}{\frac{d}{d\theta}\left\{10\log_{10}\left[\frac{f_1(\theta)}{f_o(\theta)}\right]^2\right\}},$$

where

$[f_1(\theta)]^2$ = power pattern of high beam in absence of multipath;

$[f_o(\theta)]^2$ = power pattern of low beam in absence of multipath;

$M_1[f_1(\theta)]^2 = [f_1(\theta)]^2 + 2\rho D f_1(\theta) f_1(-\theta'') \cos\psi_1 + \rho^2 D^2 [f_1(-\theta'')]^2$

= total power received on high beam;

$M_o[f_o(\theta)]^2 = [f_o(\theta)]^2 + 2\rho D f_o(\theta) f_o(-\theta'') \cos\psi_o + \rho^2 D^2 [f_o(-\theta'')]^2$

= total power received on low beam;

Table 1. Summary of Important Results for Target at Elevation (Cont.)

$$\left(\frac{S}{N}\right)_1 = \frac{1}{2}\left[\frac{f_1(\theta)}{\sigma_n}\right]^2$$

= signal to noise ratio on high beam in the absence of multipath (not in dB);

$$\left(\frac{S}{N}\right)_{M1} = \frac{M_1 f_1^2}{2\sigma_n^2} = \frac{[f_1(\theta)]^2 + 2\rho\, D f_1(\theta) f_1(-\theta'') \cos\psi_1 + \rho^2 D^2 [f_1(-\theta'')]^2}{2\sigma_n^2}$$

= total signal to noise ratio in the high beam;

$$M_1 = 1 + \frac{2\rho\, D f_1(-\theta'') \cos\psi_1}{f_1(\theta)} + \left[\frac{f_1(-\theta'')}{f_1(\theta)}\right]^2 \quad ;$$

$$M_o = 1 + \frac{2\rho\, D f_o(-\theta'') \cos\psi_o}{f_o(\theta)} + \left[\frac{f_o(-\theta'')}{f_o(\theta)}\right]^2 \quad .$$

M_1 and M_o are the normalized total powers in the high and low beams, respectively.

2. THEORETICAL ANALYSIS

From Reference 2 we recall that the signal voltage received by the primary horn is proportional to

$$S_o = \mathrm{Re}\{[f_o(\theta) + \rho\, D f_o(-\theta'')\, e^{i\psi_o}]\, e^{i\omega t}\} \quad , \tag{1}$$

and the signal received by the secondary horn can be written as

$$S_1 = \mathrm{Re}\{[f_1(\theta) + \rho\, D f_1(-\theta'')\, e^{i\psi_1}]\, e^{i\omega t}\} \quad , \tag{2}$$

where the far-field pattern of the primary horn is

$$f_o(\theta) \exp[i\phi_o(\theta)] \quad , \quad \psi_o = \phi_o(-\theta'') - \phi_o(\theta) + k\Delta R + \gamma \quad ,$$

that of the secondary horn is

$$f_1(\theta) \exp[i\phi_1(\theta)] \qquad \psi_1 = \phi_1(-\theta'') - \phi_1(\theta) + k\Delta R + \gamma \quad ,$$

ΔR is the path difference between the direct and multipath links, γ is the phase angle of the earth reflection coefficient, ρ is the magnitude of the earth reflection coefficient, and D is the spherical-earth dispersion factor. The angles θ and θ' are shown in Figure 1.

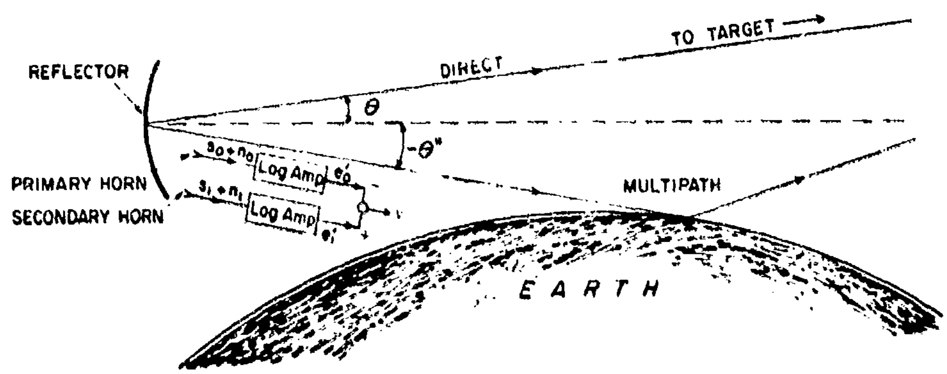

Figure 1. Multipath Geometry

The noise n_o, in the primary channel, at the input to the log-amplifier can be assumed to be a narrow-band gaussian process, and can be written as

$$n_o = x_o \cos \omega t - y_o \sin \omega t \quad ,$$

where x_o and y_o are the in-phase and quadrature noise components, respectively. Similarly the noise in the secondary channel is

$$n_1 = x_1 \cos \omega t - y_1 \sin \omega t \quad . \tag{4}$$

If we now define

$$A_o = f_o(\theta) \quad ,$$

$$B_o = \rho D f_o(-\theta'') \quad ,$$

$$A_1 = f_1(\theta) \quad ,$$

$$B_1 = \rho D f_1(-\theta'') \quad ,$$

we can write the total voltage in each channel (at the input to the log-amp) as

$$e_o = (A_o + x_o + B_o \cos \psi_o) \cos \omega t - (B_o \sin \psi_o + y_o) \sin \omega t \quad , \tag{5}$$

$$e_1 = (A_1 + x_1 + B_1 \cos \psi_1) \cos \omega t - (B_1 \sin \psi_1 + y_1) \sin \omega t \quad . \tag{6}$$

The output of each log-amp can be written as

$$e' = K \ln |e|^2 \quad ,$$

where K is a constant and $|e|^2$ is the envelope squared of e. Therefore

$$e_o' = K \ln [(A_o + x_o + B_o \cos \psi_o)^2 + (B_o \sin \psi_o + y_o)^2] \quad , \tag{7}$$

$$e_1' = K \ln [(A_1 + x_1 + B_1 \cos \psi_1)^2 + (B_1 \sin \psi_1 + y_1)^2] \quad , \tag{8}$$

and the output voltage V defined in Figure 1 is then

$$V = K \ln \left| \frac{e_1}{e_o} \right|^2$$

$$= K \ln \left(\frac{A_1}{A_o} \right)^2$$

$$+ K \ln \left\{ \frac{M_1 + \frac{2x_1}{A_1}\left(1 + \frac{B_1}{A_1} \cos \psi_1\right) + \frac{x_1^2}{A_1^2} + \frac{2B_1 y_1}{A_1^2} \sin \psi_1 + \frac{y_1^2}{A_1^2}}{M_o + \frac{2x_o}{A_o}\left(1 + \frac{B_o}{A_o} \cos \psi_o\right) + \frac{x_o^2}{A_o^2} + \frac{2B_o y_o}{A_o^2} \sin \psi_o + \frac{y_o^2}{A_o^2}} \right\} \tag{9}$$

where

$$M_1 = 1 + \frac{2B_1}{A_1}\cos\psi_1 + \left(\frac{B_1}{A_1}\right)^2 \quad . \tag{10}$$

$$M_o = 1 + \frac{2B_o}{A_o}\cos\psi_o + \left(\frac{B_o}{A_o}\right)^2 \quad . \tag{11}$$

The first term in (9) is precisely the voltage one would have if there were no noise and no multipath. The second term contains the combined effects of the noise and multipath. Note that because x_o, y_o, x_1, y_1 are random quantities this second term is a random variable.

2.1 Case I: Multipath Absent

Let us first study the effect of noise on the received voltage V. If multipath is absent we have $B_o = B_1 = 0$ and (9) becomes

$$V = V_o + K \ln \left[\frac{1 + \frac{2x_1}{A_1} + \frac{(x_1^2 + y_1^2)}{A_1^2}}{1 + \frac{2x_o}{A_o} + \frac{(x_o^2 + y_o^2)}{A_o^2}} \right] \quad . \tag{12}$$

where

$$V_o = K \ln \left(\frac{A_1}{A_o}\right)^2 \quad . \tag{13}$$

Note that V_o is the voltage which corresponds to the actual target altitude. That is, the target elevation θ is

$$\theta = F^{-1}(V_o) \tag{14}$$

(see Eqs. (1) and (2) of Reference 2).

If the signal to noise ratio is large, the second term in (12) can be expanded in a Taylor series in

$$\left(\frac{x_1}{A_1}\right) \ll 1 \quad , \quad \left(\frac{y_1}{A_1}\right) \ll 1 \quad , \quad \left(\frac{x_o}{A_o}\right) \ll 1 \quad , \quad \text{and so forth.}$$

If we use the fact that

$$\langle x_o \rangle = \langle y_o \rangle = \langle x_1 \rangle = \langle y_1 \rangle = 0 \quad , \tag{15}$$

$$\langle x_1 y_o \rangle = \langle x_o y_1 \rangle = \langle x_1 y_1 \rangle = \langle x_o y_o \rangle = 0 \quad , \tag{16}$$

$$\langle x_o^2 \rangle = \langle y_o^2 \rangle = \langle x_1^2 \rangle = \langle y_1^2 \rangle = \sigma_n^2 \quad , \tag{17}$$

where $\langle \; \rangle$ denotes an ensemble average, we can show that

$$\langle V \rangle = V_o \quad , \tag{18}$$

$$\sigma_V^2 = \langle V^2 \rangle - \langle V \rangle^2 = 4K^2 \sigma_n^2 \left(\frac{1}{A_1^2} + \frac{1}{A_o^2}\right) \quad . \tag{19}$$

Therefore, with noise only present, on the average the voltage measured will be the correct voltage V_o, but there will be a spread in measured values about the correct value V_o with a standard deviation

$$\sigma_V = 2K\sigma_n \left(\frac{1}{A_1^2} + \frac{1}{A_o^2}\right)^{1/2} \quad . \tag{20}$$

By using (20) we can calculate the mean square elevation error caused by this error in measured voltage. The voltage error ΔV and elevation error $\Delta\theta$ can be related via

$$\Delta V \simeq \frac{dV_o}{d\theta} \Delta\theta \quad . \tag{21}$$

Therefore

$$\langle \Delta\theta^2 \rangle = \frac{\langle \Delta V^2 \rangle}{\left(\frac{dV_o}{d\theta}\right)^2} , \tag{22}$$

or

$$\sigma_\theta = \sqrt{\langle \Delta\theta^2 \rangle} = \frac{\sigma_V}{\left(\frac{dV_o}{d\theta}\right)} . \tag{23}$$

Of course $\langle \Delta\theta \rangle = 0$ by virtue of (18). If we substitute (20) into (23), and use the fact that

$$\frac{dV_o}{d\theta} = 2K \frac{d}{d\theta} \ln\left(\frac{A_1}{A_o}\right) = 2K \frac{d}{d\theta} \ln\left[\frac{f_1(\theta)}{f_o(\theta)}\right] ,$$

we get

$$\sigma_\theta = \frac{1}{\left(\frac{A_o}{\sigma_n}\right)} \frac{\left(\frac{A_o^2}{A_1^2} + 1\right)^{1/2}}{\frac{d}{d\theta} \ln\left(\frac{A_1}{A_o}\right)} . \tag{24}$$

Finally, we define $(S/N)_o = (1/2)(A_o/\sigma_n)^2$ as the signal to noise ratio in the primary channel, at the input to the log-amp. The standard deviation in the angular error then becomes

$$\sigma_\theta = \frac{1}{\left[2\left(\frac{S}{N}\right)_o\right]^{1/2}} \frac{\left(\left[\frac{f_o(\theta)}{f_1(\theta)}\right]^2 + 1\right)^{1/2}}{\frac{d}{d\theta} \ln\left[\frac{f_1(\theta)}{f_o(\theta)}\right]} = \frac{8.7\left[1 + \left(\frac{f_o}{f_1}\right)^2\right]^{1/2}}{\left[2\left(\frac{S}{N}\right)_o\right]^{1/2} \frac{d}{d\theta}\left\{10 \log_{10}\left(\frac{f_1}{f_o}\right)^2\right\}} . \tag{25a}$$

We can also rewrite (25a) in terms of the signal to noise ratio $(S/N)_1 = (1/2)(A_1/\sigma_n)^2$ in the secondary channel. We get

$$\sigma_\theta = \frac{8.7\left[1+\left(\frac{f_1}{f_o}\right)^2\right]^{1/2}}{\left[2\left(\frac{S}{N}\right)_1\right]^{1/2}\frac{d}{d\theta}\left\{10\log_{10}\left(\frac{f_1}{f_o}\right)^2\right\}} . \tag{25b}$$

2.2 Case 2: Multipath Present but M_1 and M_o Never Zero

Let us now consider the effect of both noise and multipath for the case when both ψ_o and ψ_1 are not near $(2M + 1)\pi$. In that case, even if A_1 and B_1 are nearly equal [that is, $\rho D f_1(-\theta'') \simeq f_1(\theta)$], the quantities

$$A_1^2 M_1 = A_1^2 + 2B_1 A_1 \cos\psi_1 + B_1^2$$

and

$$A_o^2 M_o = A_o^2 + 2B_o A_o \cos\psi_o + B_o^2$$

are never close to zero. We next rewrite (9) as

$$V = V_o + V_m + K\ln\left(\frac{1+\frac{2x_1}{M_1A_1}\left(1+\frac{B_1}{A_1}\cos\psi_1\right)+\frac{x_1^2}{M_1A_1^2}+\frac{2B_1y_1}{M_1A_1^2}\sin\psi_1+\frac{y_1^2}{M_1A_1^2}}{1+\frac{2x_o}{M_oA_o}\left(1+\frac{B_o}{A_o}\cos\psi_o\right)+\frac{x_o^2}{M_oA_o^2}+\frac{2B_oy_o}{M_oA_o^2}\sin\psi_o+\frac{y_o^2}{M_oA_o^2}}\right) , \tag{26}$$

where

$$V_m = K\ln\left(\frac{M_1}{M_o}\right) . \tag{27}$$

Note that V_m is the error due to multipath alone; this is the error we would have in the absence of noise. We next assume that the noise on each channel is small compared with the total received signal (direct plus multipath). That is

$$\frac{\langle x_1^2 \rangle}{M_1 A_1^2} \ll 1 \quad , \quad \frac{\langle x_o^2 \rangle}{M_o A_o^2} \ll 1 \quad , \quad \text{and so forth.} \tag{28}$$

We have found by numerous numerical calculations that (28) almost always is valid in cases of interest. In this case we may expany the third term in (26) in a Taylor series. After considerable manipulation, and the extensive use of (15) through (17), we find

$$\langle V \rangle = V_o + V_m \quad , \tag{29}$$

$$\sigma_V^2 = \langle V^2 \rangle - \langle V \rangle^2 = 4K^2 \sigma_n^2 \left[\frac{1}{M_1 A_1^2} + \frac{1}{M_o A_o^2} \right] . \tag{30}$$

That is, in this limit the error in the average voltage is caused only by the multipath term V_m. However, there is a spread about the average voltage with a standard deviation

$$\sigma_V = 2K\sigma_n \left(\frac{1}{M_1 A_1^2} + \frac{1}{M_o A_o^2} \right)^{1/2} . \tag{31}$$

Upon comparing (31) with (20) we see that when multipath is present we replace the direct signals A_1^2 and A_o^2 by the total (direct plus multipath) signals

$$M_1 A_1^2 = A_1^2 + 2A_1 B_1 \cos \psi_1 + B_1^2 \quad \text{and} \quad M_o A_o^2 = A_o^2 + 2A_o B_o \cos \psi_o + B_o^2 \quad .$$

By using (29) and (30) along with (21) we can readily obtain the mean elevation error and its standard deviation. We get , since $\Delta V = \langle V \rangle - V_o = V_o + V_m - V_o$

$$\langle \Delta\theta \rangle = \frac{\ln\left(\frac{M_1}{M_o}\right)}{2 \frac{d}{d\theta} \ln\left(\frac{A_1}{A_o}\right)} , \tag{32}$$

$$\sigma_\theta = \sqrt{\langle \Delta\theta^2 \rangle} \simeq \frac{\sigma_n \left(\dfrac{1}{M_1 A_1^2} + \dfrac{1}{M_o A_o^2} \right)^{1/2}}{\dfrac{d}{d\theta} \ln \left(\dfrac{A_1}{A_o} \right)} \quad . \tag{33a}$$

If we define $(S/N)_{Mo} = M_o A_o^2 / 2\sigma_n^2$ as the signal to noise ratio in the primary channel, including multipath, and

$$\left(\frac{S}{N}\right)_{M1} = \frac{M_1 A_1^2}{2\sigma_n^2}$$

as S/N in the secondary channel, then (33a) can be rewritten as

$$\sigma_\theta = \frac{1}{\left[2\left(\frac{S}{N}\right)_{Mo} \right]^{1/2}} \frac{\left(\dfrac{M_o A_o^2}{M_1 A_1^2} + 1 \right)^{1/2}}{\dfrac{d}{d\theta} \ln \left(\dfrac{A_1}{A_o} \right)} = \frac{1}{\left[2\left(\frac{S}{N}\right)_{M1} \right]^{1/2}} \frac{\left(1 + \dfrac{M_1 A_1^2}{M_o A_o^2} \right)^{1/2}}{\dfrac{d}{d\theta} \ln \left(\dfrac{A_1}{A_o} \right)} \quad . \tag{33b}$$

Therefore, there is an average elevation error caused strictly by the multipath, and a fluctuation about that average caused by a combination of both noise and multipath.

2.3 Case 3: Multipath Present and Either M_1 or M_o Near Zero

Let us consider the case when $A_o \approx B_o$ and $\psi_o \approx \pi$, but $\psi_1 \neq \pi$.* (The case when $A_1 \approx B_1$ and $\psi_1 \approx \pi$ is equivalent.) The condition $A_o \approx B_o$ will occur for situations where the earth reflectivity is nearly unity; this may be the case for small grazing-angle situations. For the small grazing-angle case we may have $\rho D \approx 1$ and $-\theta'' \approx \theta$, so that $f_o(-\theta'') \approx f(\theta)$.

---

*We can have $\psi_o \approx \pi$ but $\psi_1 \neq \pi$ because $\phi_o(-\theta'') - \phi_o(\theta)$ is not necessarily equal to $\phi_1(-\theta'') - \phi_1(\theta)$.

When $A_o \simeq B_o$ and $\psi_o = \pi$ we may approximate (9) as

$$V \simeq K \ln \left\{ \frac{M_1 A_1^2 \left[1 + \frac{2x_1}{M_1 A_1} + \frac{2x_1 B_1}{M_1 A_1^2} \cos \psi_1 + \frac{x_1^2}{M_1 A_1^2} + \frac{2B_1 y_1}{M_1 A_1^2} \sin \psi_1 + \frac{y_1^2}{M_1 A_1^2} \right]}{x_o^2 + y_o^2} \right\} . \quad (34)$$

In writing (34) we have assumed that the net primary-channel signal

$$A_o^2 + 2A_o B_o \cos \psi_o + B_o^2 \simeq (A_o - B_o)^2$$

is so small that it is exceeded by the noise $x_o^2 + y_o^2$. If we again assume that the signal to noise ratio in the secondary channel is large it is possible to expand the numerator in (34) in a Taylor series. The result for the ensemble average of V then is

$$\langle V \rangle = K \ln (M_1 A_1^2) - K \langle \ln (x_o^2 + y_o^2) \rangle \quad . \quad (35)$$

In order to evaluate $\langle \ln (x_o^2 + y_o^2) \rangle$ we use the fact that x_o and y_o have a joint probability density

$$p(x_o, y_o) = \frac{1}{2\pi\sigma_n^2} \exp \left\{ - \frac{x_o^2 + y_o^2}{2\sigma_n^2} \right\} \quad . \quad (36)$$

Then

$$\langle \ln(x_o^2 + y_o^2) \rangle = \frac{1}{2\pi\sigma_n^2} \iint_{-\infty}^{\infty} dx\, dy \ln (x^2 + y^2)\, e^{-(x^2+y^2)/2\sigma_n^2}$$

$$= \frac{1}{\sigma_n^2} \int_0^{\infty} r\, dr \ln (r^2)\, e^{-(r^2/2\sigma_n^2)}$$

$$= \frac{1}{2\sigma_n^2} \int_0^{\infty} dt \ln t\, e^{-(t/2\sigma_n^2)} \quad (37a)$$

$$\langle \ln(x_o^2 + y_o^2)\rangle = -C + \ln(2\sigma_n^2) \quad , \tag{37b}$$

where C = Catalan's constant = 0.577215. The integral in (37a) is from (4.331) in Gradshteyn and Ryzhik.[4] Upon using (37b) in (35) we get

$$\frac{\langle V\rangle}{K} = \ln\left(\frac{A_1^2 M_1}{2\sigma_n^2}\right) + C \quad . \tag{38}$$

A similar calculation yields

$$\sigma_V^2 = \langle V^2\rangle - \langle V\rangle^2 = K^2\left(\frac{4\sigma_n^2}{M_1 A_1^2} + \frac{\pi^2}{6}\right) \quad . \tag{39}$$

In deriving (39) we have used the relation

$$\begin{aligned} \langle \ln^2(x^2 + y^2)\rangle &= \frac{1}{2\sigma_n^2}\int_0^\infty dt \ln^2 t \exp\left(-\frac{t}{2\sigma_n^2}\right) \\ &= \frac{\pi^2}{6} + \left((C - \ln(2\sigma_n^2)\right)^2 \quad , \end{aligned} \tag{40}$$

where the above integral is from (4.335) in Gradshteyn and Ryzhik.

By using (38) and (13) we can calculate the mean elevation error. We recall from (13) that for a target at an elevation angle θ the correct voltage we should measure (in the absence of noise and multipath, which produce errors) is

$$V_o = K \ln\left(\frac{A_1}{A_o}\right)^2 \quad . \tag{41}$$

4. Gradshteyn, I, and Ryzhik, I. (1965) Tables of Integrals, Series, and Products, Academic Press (New York).

Therefore the voltage error ΔV is

$$\begin{aligned}\Delta V &= \langle V \rangle - V_o \\ &= K \ln\left(\frac{M_1 A_1^2}{2\sigma_n^2}\right) + KC - K \ln\left(\frac{A_1^2}{A_o^2}\right) \\ &= K\left[\ln\left(\frac{M_1 A_o^2}{2\sigma_n^2}\right) + C\right] \ .\end{aligned} \tag{42}$$

Therefore using (21) we find that the mean elevation error is

$$\langle \Delta\theta \rangle = \frac{\ln\left(\dfrac{M_1 A_o^2}{2\sigma_n^2}\right) + C}{2\dfrac{d}{d\theta}\ln\left(\dfrac{A_1}{A_o}\right)} \tag{43}$$

and the fluctuation about that mean is

$$\sigma_\theta = [\langle \Delta\theta^2 \rangle]^{1/2} = \frac{\left[\dfrac{\pi^2}{6} + \dfrac{4\sigma_n^2}{M_1 A_1^2}\right]^{1/2}}{2\dfrac{d}{d\theta}\ln\left(\dfrac{A_1}{A_o}\right)} \ . \tag{44}$$

For the case when the signal in the secondary channel is negligible (that is, $A_1 \simeq B_1$ and $\psi_1 \simeq \pi$) but that in the primary channel is not, we find

$$\langle \Delta\theta \rangle \simeq \frac{-\ln\left(\dfrac{M_o A_1^2}{2\sigma_n^2}\right) + C}{2\dfrac{d}{d\theta}\left(\ln\dfrac{A_1}{A_o}\right)} \ , \tag{45}$$

$$\sigma_\theta = \frac{\left[\frac{\pi^2}{6} + \frac{4\sigma_n^2}{M_o A_o^2}\right]^{1/2}}{2\frac{d}{d\theta}\ln\left(\frac{A_1}{A_o}\right)} \quad . \tag{46}$$

2.4 Case 4: Zero Net Signal in Both Channels

If it happens that

$$\phi_o(-\theta'') - \phi_o(\theta) = \phi_1(-\theta'') - \phi_1(\theta) \quad ,$$

so that both ψ_o and ψ_1 can simultaneously equal $(2M + 1)\pi$, where M = integer, it is possible that the net signal in both channels may simultaneously equal zero (assuming of course $A_o \simeq B_o$, $A_1 \simeq B_1$). In that case

$$V \simeq K \ln\left(\frac{x_1^2 + y_1^2}{x_o^2 + y_o^2}\right) \quad , \tag{47}$$

and

$$\langle V \rangle \simeq 0 \quad , \tag{48}$$

$$\sigma_V^2 = \langle V^2 \rangle - \langle V \rangle^2 \approx 0 \quad . \tag{49}$$

In this case the voltage error

$$\Delta V = \langle V \rangle - V_o = -V_o \tag{50}$$

and the mean elevation error is

$$\langle \Delta\theta \rangle = \frac{-V_o}{2\frac{d}{d\theta}\ln\left(\frac{A_1}{A_o}\right)} = \frac{-\ln\left(\frac{A_1}{A_o}\right)^2}{\frac{d}{d\theta}\ln\left(\frac{A_1}{A_o}\right)^2} \quad . \tag{51}$$

and of course

$$\langle \Delta\theta^2 \rangle = 0 \quad .$$

3. SAMPLE CALCULATIONS

Let us consider the elevation error caused by noise alone for the configuration denoted by Case A in Reference 2. Let $f_o(\theta)$ denote the signal received in the low beam and $f_1(\theta)$ denote the signal received in the high beam. Because, for small target elevations, $f_1/f_o \ll 1$, it is convenient to use (25b) for the elevation error, or

$$\sigma_\theta = \frac{8.7\left[1+\left(\frac{f_1}{f_o}\right)^2\right]^{1/2}}{\left[2\left(\frac{S}{N}\right)_1\right]^{1/2}\frac{d}{d\theta}\left\{10\log_{10}\left(\frac{f_1}{f_o}\right)^2\right\}}$$

$$\approx \frac{8.7}{\left[2\left(\frac{S}{N}\right)_1\right]^{1/2}\frac{d}{d\theta}\left\{10\log_{10}\left(\frac{f_1}{f_o}\right)^2\right\}} , \tag{53}$$

where $(S/N)_1 = (1/2)(f_1/\sigma_n)^2$ is the signal to noise ratio for the high beam. For a target at elevation $\theta = 0^o$ we find from Figures A.1 and B.1 in Reference 2 that $d/d\theta\,\{10\log_{10}(f_1/f_o)^2\} = 3.5/\text{deg}$, so that $\sigma_\theta \sim 2.5\,(2S/N)_1^{-1/2}$. Therefore, in order to have an elevation error of 3000 ft when the target range is 100 miles we require $\sigma_\theta = 0.326^o$. This leads to a high-beam signal to noise ratio of 14.7 dB.

When multipath is present we get a different requirement on $(S/N)_1$ because now $|\langle\Delta\theta\rangle| + \sigma_\theta$ must be less than 0.326^o. If neither M_1 nor M_o is too small we may use (32) and (33b) to obtain

$$\text{Max Angular Error} \approx |\langle\Delta\theta\rangle| + \sigma_\theta$$

$$= \frac{4.35\left|\ln\left(\frac{M_1}{M_o}\right)\right| + 8.7\left(\frac{2S}{N}\right)_{M1}^{-1/2}\left(1+\frac{M_1A_1^2}{M_oA_o^2}\right)^{1/2}}{\frac{d}{d\theta}\left\{10\log_{10}\left(\frac{A_1}{A_o}\right)^2\right\}} . \tag{54}$$

If we again consider the configuration denoted by Case A in Reference 2 and again require a maximum angular error of 0.326° for a target at elevation $\theta = 0^{\circ}$, we now get the requirement that

$$\left|\ln\left(\frac{M_1}{M_o}\right)\right| + 2\left(\frac{2S}{N}\right)_{M1}^{-1/2}\left[1 + \frac{M_1 f_1^2(\theta)}{M_o f_o^2(\theta)}\right]^{1/2} \leq 0.263 \quad , \tag{55}$$

where

$$M_1 = 1 + \frac{2\rho D f_1(-\theta'')}{f_1(\theta)}\cos\psi_1 + \left[\frac{\rho D f_1(-\theta'')}{f_1(\theta)}\right]^2$$

$$M_o = 1 + \frac{2\rho D f_o(-\theta'')}{f_o(\theta)}\cos\psi_o + \left[\frac{\rho D f_o(-\theta'')}{f_o(\theta)}\right]^2 \quad .$$

We may assume that the total signal $M_1\ f_1^2$ in the high beam is much less than the total signal $M_o\ f_o^2$ in the low beam; this leads to the requirement that

$$\left|\ln\left(\frac{M_1}{M_o}\right)\right| + \frac{2}{\left[2\left(\frac{S}{N}\right)_{M1}\right]^{1/2}} \leq 0.263 \quad . \tag{56}$$

Unfortunately, for the patterns in case A of Reference 2 it turns out that usually $|\ln(M_1/M_o)| > 0.263$. Therefore for this case there is no signal to noise ratio which makes the angular error less than 0.263°.

4. EFFECT OF CLUTTER

In general we can write the clutter return in the primary and secondary channels as complex phasors $c_o = r_o \exp(i\eta_o)$ and $c_1 = r_1 \exp(i\eta_1)$, respectively. The net clutter return in each channel is proportional to the integral of the radiation pattern of each antenna over the complex clutter distribution $h(x, y)$. That is

$$c_0 = r_0 e^{i\eta_0} = \text{const} \iint_E dS\, \hat{f}_0(x, y)\, h(x, y) \quad , \tag{57}$$

$$c_1 = r_1 e^{i\eta_1} = \text{const} \iint_E dS\, \hat{f}_1(x, y)\, h(x, y) \quad , \tag{58}$$

where dS is the element of surface area, $\hat{f}_0(x, y) = f_0[\theta(x, y)]\, e^{i\phi_0(x, y)}$ and $f_1(x, y) = f_1[\theta(x, y)]\, e^{i\phi_1(x, y)}$ are the field patterns of the low and high beams (with θ and ϕ_0 expressed in terms of x and y), and E is the illuminated region of the earth's surface. For small grazing angles E is equal to $(c\tau/2)(R\phi_2)$ where τ is the pulse length of the radar signal, c is the speed of light, R is the clutter range and ϕ_2 is the azimuthal beamwidth. It is found experimentally[5] that for small grazing angles, r_0 and r_1 are log normally distributed random variables, and η_0 and η_1 are uniformly distributed random variables. Because the phases are uniformly distributed it is evident that $\langle c_0 \rangle = \langle c_1 \rangle = 0$; of course $\langle |c_0|^2 \rangle$ and $\langle |c_1|^2 \rangle$ are not equal to zero. The only other quantity we will have to calculate is $\langle c_0 c_1^* \rangle$; unfortunately this is not zero, because from (57) and (58) we have

$$\langle c_0 c_1^* \rangle = \text{const} \iiiint dS\, dS'\, f_0(x, y)\, f_1^*(x', y') \langle h(x, y)\, h^*(x', y') \rangle$$

and the correlation function

$$R(x, y; x', y') \equiv \langle h(x, y)\, h^*(x', y') \rangle$$

is not zero.[6] Therefore in our calculations we will not, in general, be able to ignore $\langle c_0 c_1^* \rangle$.

We are now in a position to discuss how clutter is included in our analysis. When clutter is included, (5) and (6) become

5. Nathanson, F. (1969) Radar Design Principles, McGraw-Hill, New York.
6. Beckmann, P. and Spizzichino, A. (1963) The Scattering of EM Waves From Rough Surfaces, MacMillan, New York.

$$e_o = (A_o + x_o + B_o \cos \psi_o + r_o \cos \eta_o) \cos \omega t$$

$$- (B_o \sin \psi_o + y_o + r_o \sin \eta_o) \sin \omega t \quad . \tag{59}$$

$$e_1 = (A_1 + x_1 + B_1 \cos \psi_1 + r_1 \cos \eta_1) \cos \omega t$$

$$- (B_1 \sin \psi_1 + y_1 + r_1 \sin \eta_1) \sin \omega t \quad . \tag{60}$$

Therefore, in our previous analysis we simply replace x_o by $x_o + r_o \cos \eta_o$, y_o by $y_o + r_o \sin \eta_o$, x_1 by $x_1 + r_1 \cos \eta_1$, and y_1 by $y_1 + r_1 \sin \eta_1$. If we do this and recall that the noise and clutter are uncorrelated, and each has a zero mean so that $\langle x_o r_o \cos \eta_o \rangle = \langle x_o \rangle \langle r_o \cos \eta_o \rangle = 0$, $\langle y_o r_o \sin \eta_o \rangle = 0$, $\langle x_1 r_1 \cos \eta_1 \rangle = 0$, $\langle y_1 r_1 \sin \eta_1 \rangle = 0$, we find that (29) is unchanged but (30) is altered by the clutter. That is

$$\langle V \rangle = V_o + V_M \quad . \tag{61}$$

but

$$\frac{\sigma_V^2}{K^2} = 4\sigma_n^2 \left(\frac{1}{M_1 A_1^2} + \frac{1}{M_o A_o^2} \right) + 2 \left\{ \frac{\langle |c_1|^2 \rangle}{M_1 A_1^2} + \frac{\langle |c_o|^2 \rangle}{M_o A_o^2} \right\}$$

$$- \frac{8 \langle r_1 r_o \rangle}{M_1 A_1 M_o A_o} \left\{ \langle \cos \eta_o \cos \eta_1 \rangle \left(1 + \frac{B_1}{A_1} \cos \psi_1 \right) \left(1 + \frac{B_o}{A_o} \cos \psi_o \right) \right.$$

$$+ \langle \sin \eta_o \cos \eta_1 \rangle \left(\frac{B_o}{A_o} \sin \psi_o \right) \left(1 + \frac{B_1}{A_1} \cos \psi_1 \right)$$

$$+ \langle \sin \eta_1 \cos \eta_o \rangle \left(\frac{B_1}{A_1} \sin \psi_1 \right) \left(1 + \frac{B_o}{A_o} \cos \psi_o \right)$$

$$\left. + \langle \sin \eta_o \sin \eta_1 \rangle \left(\frac{B_1}{A_1} \sin \psi_1 \right) \left(\frac{B_o}{A_o} \sin \psi_o \right) \right\} \quad . \tag{62}$$

In deriving (62) we have assumed the signal to clutter ratio is large. Also $\langle |c_1|^2 \rangle = \langle r_1^2 \rangle$ and $\langle |c_o|^2 \rangle = \langle r_o^2 \rangle$. In order to simplify (62) we shall assume that the phase of the low beam radiation pattern $\hat{f}_o(x, y)$ is equal to the phase of the high beam pattern $\hat{f}_1(x, y)$. Then from (57) and (58) it is clear that η_o must equal η_1. Because η_o and η_1 are uniformly distributed we then have that $\langle \cos \eta_o \cos \eta_1 \rangle = \langle \sin \eta_o \sin \eta_1 \rangle = 1/2$, and $\langle \cos \eta_o \sin \eta_1 \rangle = \langle \sin \eta_o \cos \eta_1 \rangle = 0$ so that

$$\frac{\sigma_V^2}{K^2} = 4\sigma_n^2 \left(\frac{1}{M_1 A_1^2} + \frac{1}{M_o A_o^2} \right) + 2 \left\{ \frac{\langle |c_1|^2 \rangle}{M_1 A_1^2} + \frac{\langle |c_o|^2 \rangle}{M_o A_o^2} \right\}$$

$$- \frac{4 \langle r_o r_1 \rangle}{M_1 A_1 M_o A_o} \left[\left(1 + \frac{B_1}{A_1} \cos \psi_1 \right) \left(1 + \frac{B_o}{A_o} \cos \psi_o \right) + \left(\frac{B_1}{A_1} \right) \left(\frac{B_o}{A_o} \right) \right.$$

$$\left. \sin \psi_o \sin \psi_1 \right] . \tag{62a}$$

We will now try to approximate (62) and (62a) further. When we are interested in the angular error for the air search radar in Case A in Reference 2, it is clear that the total high-beam signal $M_1 A_1^2$ is much smaller than the total low-beam signal $M_o A_o^2$. Therefore in the first term in (62) and (62a) we may neglect $(M_o A_o^2)^{-1}$ in comparison with $(M_1 A_1^2)^{-1}$. Next let us examine the clutter contributions. If we use the results in Nathanson,[5] it is found that, for the low beam, the clutter to signal ratio is (ignoring sidelobe clutter because this is quite small compared with the main lobe clutter)

$$\frac{\langle |c_o|^2 \rangle}{A_o^2} = \frac{F_o^2(\theta_c) \left(\frac{c\tau}{2} \right) \phi_2 R \sigma_c I}{F_o^2(\theta_T) L_p \sigma_T} , \tag{63}$$

where $F_o(\theta_c)/F_o(\theta_T)$ is the ratio of the lower-beam power in the direction of the clutter patch to that in the direction of the target, c is the speed of light, τ is the radar pulse length, ϕ_2 is the azimuthal beamwidth, R is the target range, L_p is the beam shape loss (usually taken as 1.6 dB), σ_T is the target cross section, σ_c is the reflectivity per unit cross section of clutter and I is the clutter improvement factor of the MTI. Similarly, for the high beam we have (here the sidelobe clutter is significant, but still less than the main-beam clutter, provided the high-beam peak occurs at angles less than 7°)

$$\frac{\langle |c_1|^2\rangle}{A_1^2} = \frac{F_1^2(\theta_c)}{F_1^2(\theta_T)} \; \frac{\left(\frac{c\tau}{2}\right) R\sigma_c I\phi_2}{L_p\,\sigma_T} , \tag{64}$$

where $F_1(\theta_c)/F_1(\theta_T)$ is the ratio of the high-beam power in the direction of the clutter patch to that in the direction of the target. For the air search radar in Case A of Reference 2 we have for a target at 1.5^o, and assuming the angle θ_c of the ground clutter is approximately 0^o, that

$$\frac{F_o(\theta_c)}{F_o(\theta_T)} \simeq 0.71 \quad , \quad \frac{F_1(\theta_c)}{F_1(\theta_T)} \simeq 0.17 \quad .$$

If we use these results in (63) and (64) we find that

$$\xi = \frac{\langle |c_1|^2\rangle / A_1^2}{\langle |c_o|^2\rangle / A_o^2} \simeq 0.058 \quad .$$

Therefore in (62) and (62a) we can neglect $\langle |c_1|^2\rangle/(M_1A_1^2)$ in comparison with $\langle |c_o|^2\rangle/(M_oA_o^2)$. (Note that M_o is the same order as M_1, as is evident from (10) and (11).)

In the last term in (62a) we have that

$$\langle r_o r_1\rangle \leq \left[\langle r_o^2\rangle\langle r_1^2\rangle\right]^{1/2} = \left[\langle |c_o|^2\rangle\langle |c_1|^2\rangle\right]^{1/2} \quad .$$

Therefore the ratio of the last to the second term in (62a) is at most

$$\frac{\text{last term}}{\text{second term}} \leq 2\xi^{1/2}\left[\frac{\left(1+\frac{B_1}{A_1}\cos\psi_1\right)\left(1+\frac{B_o}{A_o}\cos\psi_o\right)+\left(\frac{B_1}{A_1}\right)\left(\frac{B_o}{A_o}\right)\sin\psi_o\,\sin\psi_1}{M_1}\right]$$

$$\leq 0.48 \quad ,$$

because the term in square brackets is less than unity. Therefore we shall neglect the last term in (62) and approximate the voltage error by

$$\sigma_V^2 \simeq 4K^2 \left[\frac{\sigma_n^2}{M_1 A_1^2} + \frac{\sigma_o^2}{M_o A_o^2} \right] , \tag{65}$$

where

$$\sigma_o^2 = \frac{1}{2}\langle |c_o|^2 \rangle = \frac{1}{2}\langle r_o^2 \rangle \quad .$$

From (61) and (65) we now have

$$\langle \Delta\theta \rangle = \frac{\ln\left(\frac{M_1}{M_o}\right)}{2\frac{d}{d\theta}\ln\left(\frac{A_1}{A_o}\right)} , \tag{66}$$

$$\sigma_\theta \simeq \frac{1}{\left[2\left(\frac{S}{N}\right)_{M1}\right]^{1/2}} \frac{\left[1 + \left(\frac{M_1 A_1^2}{M_o A_o^2}\right)\left(\frac{\sigma_o}{\sigma_n}\right)^2\right]^{1/2}}{\frac{d}{d\theta}\ln\left(\frac{A_1}{A_o}\right)} , \tag{67}$$

where

$\left(\frac{S}{N}\right)_{M1} = \frac{1}{2}\frac{M_1 A_1^2}{\sigma_n^2}$ = signal to noise ratio of high beam (including multipath),

$\left(\frac{M_1 A_1^2}{M_o A_o^2}\right)$ ratio of total high-beam power to total low-beam power,

$\left(\frac{\sigma_o}{\sigma_n}\right)^2$ = clutter to noise ratio in low beam.

The signal to noise ratios can readily be expressed in terms of system parameters. We have

$$\left(\frac{S}{N}\right)_{M_1} = \frac{M_1 P_t G_1^2 H_1^2(\theta_T) \lambda^2 \sigma_T L}{k T_s B (4\pi)^3 R^4} , \tag{68}$$

where M_1 is given in (10), P_t is the transmitted power, G_1 is the gain of the high beam, $H_1(\theta_T)$ is the ratio of the high-beam power density at the target angle θ_T to the peak power density, λ is the signal wavelength, L is the system loss (beam shape loss, attenuation loss, and so forth), k is Boltzmann's constant, T_s is the system noise temperature, and B is the system bandwidth (this is of order τ^{-1}). Also

$$\left(\frac{\sigma_o}{\sigma_n}\right)^2 \simeq \frac{P_t G_o^2 H_o^2(\theta_c) L \lambda^2 \sigma_c \left(\frac{c\tau}{2}\right) \phi_2 I}{k T_s B (4\pi)^3 R^3} , \tag{69}$$

where G_o is the low-beam gain, $H_o(\theta_c)$ is the ratio of the low-beam power density at the angle θ_c (of the clutter patch) to the peak power density, and all the other quantities in (69) were defined previously.

From (69) and (67) we see that if the MTI design is such that the clutter to noise ratio is of order unity then clutter can be completely ignored in calculating angular errors, because then

$$\left(\frac{M_1 A_1^2}{M_o A_o^2}\right)\left(\frac{\sigma_o}{\sigma_n}\right)^2 \ll 1 ,$$

since $M_1 A_1^2 \ll M_o A_o^2$. In this case

$$\sigma_\theta \simeq \frac{1}{\left[2\left(\frac{S}{N}\right)_{M1}\right]^{1/2}} \; \frac{1}{\frac{d}{d\theta} \ln\left(\frac{A_1}{A_o}\right)} ,$$

so that our calculations in Eqs. (54) through (56) are then appropriate.

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